**Sample Chapter 0**

# Value-at-Risk

## Theory and Practice

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This is Chapter 0 of Glyn Holton's book **Value-at-risk: Theory and Practice**. For more information, visit www.value-at-risk.net.

Chapter 0

# Preface

## 0.1. WHAT WE'RE ABOUT

A watershed in the history of value-at-risk (VaR) was the publication of JP Morgan's *RiskMetrics Technical Document*. Writing in the third edition of that document, Guldimann (1995) identified three practical methods for calculating VaR:

- the parametric method,
- the historical simulation method, and
- the structured Monte Carlo method;

and so, the "methods" approach for describing VaR was born. Explaining VaR as comprising three "methods" is simple, intuitive and direct. The "methods" approach has been widely adopted by authors of books and research papers. Variations on the three basic "methods" abound, but only one truly new "method" has been introduced since 1995. This might be termed the "quadratic method." Rouvinez (1997) ultimately published it.

For some time, I have felt that the top-down "methods" approach for explaining VaR was flawed. It is like explaining options pricing theory by presenting the Black–Scholes (1973), Merton (1973), and Black (1976) option pricing formulas. As an alternative, a bottom-up description might start with arbitrage pricing, stochastic calculus and replicating portfolios. Top-down explanations are appealing because they go directly to results, but they lead nowhere after that. Bottom-up explanations build a foundation for deeper understanding and further research. I have felt that VaR required such a foundation.

In writing this book, I faced three challenges:

- Although I perceived that VaR needed the firm foundation of a bottom-up explanation, I had no idea what form this should take. I was starting out with a blank sheet of paper.
- More fundamental than this were certain philosophical questions relating to the subjective nature of risk and how to justify attaching a number to—measuring—a subjective notion.
- Finally, I had to decide how technical to make the book.

I addressed the first challenge with hard work. I wrote and discarded hundreds of pages. Of three preliminary chapters I wrote in 1997, hardly a trace remains. Through all this work, a flexible bottom-up framework for understanding VaR gradually emerged.

To address the philosophical issues, I turned to the libraries around Boston. I read philosophical treatises by David Hume and Rudolf Carnap. I delved into the literature on subjective probabilities. I found what I was looking for in the Operationalism of Percy Bridgman. Little of this is mentioned explicitly in the following chapters, but the spirit of Operationalism permeates the entire book.

To settle on a suitable level of mathematics, I considered both the mathematics employed by sophisticated VaR measures and the level of mathematical knowledge that might reasonably be assumed of practitioners. There was somewhat of a gap between the two. The mathematics of VaR is like the proverbial river that is a mile wide but an inch deep. There is a tremendous amount of it—calculus, linear algebra, probability, statistics, time series analysis, numerical methods—but most is not particularly deep. I decided to assume basic knowledge of calculus, linear algebra, and probability, but to fill in the rest.

## 0.2. CONTENTS OVERVIEW

The book is divided into three parts. The first contains this Preface and Chapter 1. Chapter 1 introduces the bottom-up framework employed throughout the remainder of the book. It explains important notation and terminology. It is a roadmap for subsequent chapters. On its own, Chapter 1 offers a nice introduction to VaR.

Part 2 covers essential mathematics. Starting with Chapter 2, it reviews a number of useful techniques of applied mathematics. Chapter 3 covers topics in probability theory. Assuming familiarity with basic concepts, it delves into more specialized topics, such as mixed-normal distributions, principal component analysis, the Cornish–Fisher expansion and the inversion theorem. Chapter 4 discusses basic notions from classical statistics and time series analysis. The statistics is important for a subsequent discussion of the Monte Carlo method in Chapter 5.

To avoid seeming "cookbookish," I have treated the math as a stand-alone topic and, for the most part, have resisted the temptation to immediately illustrate concepts with VaR applications. This should serve readers well, since much of the math can be used in VaR measures in different ways. Most of it is invaluable for financial applications unrelated to VaR, so it is worth learning in its own right. However, Part 2 is focused. Only topics that will be relevant later in the book are covered.

Part 3 is a bottom-up explanation of how to design VaR measures. Chapter 6 discusses issues relating to the gathering and processing of historical market data. Chapter 7 covers inference procedures. Chapters 8 and 9 cover mapping procedures. Chapter 10 closes with transformation procedures. The discussion is practical, detailing how to implement scalable production VaR systems. A number of techniques are presented for the first time in book form. Among these are:

- the mathematics of quadratic transformations;
- variance reduction techniques applicable to VaR measures; and
- essential remapping techniques.

There are practical examples throughout Part 3. Some are drawn from actual VaR measures I have implemented.

Inevitably, there is much a book cannot include. General discussions of financial risk management are outside the scope of the book. Techniques that complement VaR, such as stress testing, are also excluded.

I had hoped to include a discussion of quasi-Monte Carlo methods. Time and space limitations prevented me from doing so in anything but a cursory manner, so I decided to not do so at all. Still, the book is written with quasi-Monte Carlo methods in mind. Although it does not do so explicitly, it provides plenty of guidance for anyone familiar with quasi-Monte Carlo methods to apply them with VaR measures. A practical issue is the high dimensionality of most VaR applications. Chapter 9 on remappings details techniques for reducing those dimensionalities.

Backtesting of VaR measures is an important topic that raises certain philosophical issues. If risk is subjective, what does it mean for a measure of risk to be "right" or "wrong"; "good" or "bad"; "accurate" or "inaccurate"? I felt that a treatment of backtesting should confront such philosophical issues. To avoid writing a philosophical treatise, I have saved the discussion for another day.

Some authors use the term "value-at-risk" to refer to any probabilistic measure of financial risk, so they speak of VaR measures of credit risk or VaR measures of operational risk. Other authors, including myself, are less extravagant. In this book, VaR encompasses only measures of market risk. Some of the techniques covered can be applied to other risks, but I would not call the resulting measures "VaR measures."

It has become fashionable to caution users that VaR has certain limitations. Doing so is to state the obvious. All tools have limitations. For example, screw drivers are limited by the fact that they cannot drive nails. This book describes how to design, implement, and use VaR measures. It forgoes earnest suggestions of how VaR might not be used.

## 0.3. AUDIENCE

Through my consulting and training practice, I have worked with hundreds of practitioners—risk managers, traders, regulators, managers, software developers and consultants—representing the capital, commodity, and energy markets. This exposure has helped me understand practitioners' needs, as well as the hurdles they face in understanding, implementing, and using VaR. It has also given me a clear sense of what mathematical knowledge it is reasonable to assume in my writing. The book's primary audience is practitioners who need to implement VaR systems or otherwise require an intimate knowledge of how production VaR systems (should) work. This includes auditors, consultants and regulators. It includes many financial engineers.

The book's bottom-up approach, rigor and treatment of advanced topics will appeal to researchers. My hope is that it will spur further research by clarifying the state of the art and by identifying areas that require further study.

Finally, the book is suitable for students. It offers practical mathematics they can "sink their teeth into." Exercises offer opportunities for practice. References for further reading are indicated at the ends of chapters.

## 0.4. HOW TO READ THE BOOK

Read this chapter and especially the discussion of notation. Then proceed to Chapter 1. Depending upon your math skills, some of the quantitative examples may be a bit intimidating. Skim them and move on. You can return later.

Part 2 covers essential mathematics that is anticipated in Part 1 and used extensively in Part 3. If you need to review basic calculus, linear algebra, or probability, there are references at the ends of Chapters 2 and 3. It is not essential that you master all of Part 2 before proceeding to Part 3. Some readers may want to skim the material and refer back to it as necessary.

Don't attempt Part 3 until you have mastered Chapter 1. If you have difficulty with the mathematics of Chapter 1, learn that mathematics in Part 2, and then reread Chapter 1 before proceeding to Part 3. Part 3 has a lot of technical depth. You may want to read it twice. Go quickly through the material the first time to get an overview of how it all fits together. Read it more carefully the second time to gain deeper understanding.

Exercises are an essential part of the text. I encourage readers to work through as many as time permits. Doing so will accelerate learning and provide insights that are difficult to achieve through reading alone. Most exercises can be performed with pencil and paper or a spreadsheet. For a few, more sophisticated analytical software will be useful. Such exercises are indicated with a symbol 🖩. I had hoped to provide solutions to all exercises. Space limitations made it infeasible to do so in the book itself, so I have posted complete solutions on the Internet at

http://www.contingencyanalysis.com

There are various ways the book can be used in a classroom setting. If students have strong quantitative skills, the book can be covered in a single semester. Focus on Parts 1 and 3. Students can refer to the mathematics of Part 2 on their own as needed.

If students need to develop their quantitative skills, you might design a two-semester financial mathematics course that uses VaR to motivate topics. Cover Part 1 carefully, referring forward to the mathematics of Part 2 as necessary. Then cover highlights of Part 2. Finally, go through the chapters of Part 3 sequentially, referring back to mathematics in Part 2 as needed. You can design the course so that all the material of Part 2 is covered at some point, with much of it motivated by specific applications from Parts 1 and 3.

## 0.5. NOTATION AND TERMINOLOGY

Currencies are indicated with standard codes. Where applicable, millions are indicated as MM. For example, 3.5 million Japanese yen is indicated: JPY 3.5MM. Currency codes used in the book are shown in Exhibit 0.1.

| Code | Currency | Code | Currency |
|---|---|---|---|
| AUD | Australian dollar | JPY | Japanese yen |
| CAD | Canadian dollar | NOK | Norwegian krone |
| CHF | Swiss franc | PHP | Philippine peso |
| DKK | Danish krone | SEK | Swedish krona |
| EUR | European euro | SGD | Singapore dollar |
| GBP | British pound | THB | Thailand bath |
| GRD | Greek drachma | TWD | Taiwan dolla |
| HKD | Hong Kong dollar | USD | United States dollar |
| IDR | Indonesian rupiah | | |

**Exhibit 0.1** Currency codes.

Exchange rates are indicated as fractions, so an exchange rate of 1.62 USD/GBP indicates that one British pound is worth 1.62 US dollars. Acronyms used include those shown in Exhibit 0.2.

| | |
|---|---|
| BBA | British Bankers Association |
| CAD | Capital Adequacy Directive |
| CBOT | Chicago Board Of Trade |
| CDF | cumulative distribution function (distribution function) |
| CME | Chicago Mercantile Exchange |
| CSCE | Coffee, Sugar and Cocoa Exchange |
| EICG | explicit inversive congruential generator |
| ETL | expected tail loss |
| ICG | inversive congruential generator |
| IID | independent and identically distributed |
| IPE | International Petroleum Exchange |
| LCG | linear congruential generator |
| Libor | London Interbank Offered Rate |
| LIFFE | London International Financial Futures and Options Exchange |
| LME | London Metals Exchange |
| MGF | moment generating function |
| ML | maximum likelihood |
| MRG | multiple recursive generator |
| MSE | mean squared error |
| NYBOT | New York Board of Trade |
| NYMEX | New York Mercantile Exchange |
| NYSE | New York Stock Exchange |
| OTC | over the counter |
| P&L | profit and loss |
| PDF | probability density function |
| PF | probability function |
| RAROC | risk-adjusted return on capital |
| RORAC | return on risk-adjusted capital |
| SEC | Securities and Exchange Commission |
| TSE | Toronto Stock Exchange |
| UNCR | Uniform Net Capital Rule |
| VaR | Value-at-Risk |
| WCE | Winnipeg Commodities Exchange |

**Exhibit 0.2** Acronyms.

VaR draws on many branches of mathematics. Each offers its own notation conventions. Because these conflict, it is impossible to observe them all simultaneously. I have developed a system of notation that consistently presents financial concepts related to calculus, linear algebra, probability, statistics, time series analysis, and numerical methods. It draws on existing conventions as much as possible. It is employed uniformly throughout the book.

Random quantities are indicated with capital English letters. If they are random variables—that is, univariate—they are italic nonbold: $Q$, $R$, $S$, $X$, etc. If they are multivariate in some sense—random vectors, random matrices, stochastic processes—they are italic bold: $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S}, \boldsymbol{X}$, etc. Nonrandom quantities are indicated with lowercase italic letters. These are nonbold for scalars: $q, r, s, x$, etc. They are bold for vectors, matrices, or time series: $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{x}$, etc.[1]

[1] An exception to this notation is the (nonrandom) identity matrix. Bowing to convention, I denote it $\boldsymbol{I}$.

With this notation, if a random variable is denoted $X$, a specific realization of that random variable may be denoted $x$. Such notational correspondence between random quantities and realizations of those random quantities is employed throughout the book.

Components of vectors or matrices are distinguished with subscripts. Consider the random vector

$$\boldsymbol{Q} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}, \qquad [0.1]$$

or the matrix

$$\boldsymbol{c} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}. \qquad [0.2]$$

Time also enters the equation. To avoid confusion, I do not indicate time with subscripts. Instead, I use superscripts that precede the rest of the symbol. For example, the AUD Libor curve evolves over time. We may represent its value at time $t$ as

$$^{t}\boldsymbol{R} = \begin{pmatrix} ^{t}R_1 \\ ^{t}R_2 \\ ^{t}R_3 \\ ^{t}R_4 \\ ^{t}R_5 \\ \vdots \\ ^{t}R_{15} \end{pmatrix} \sim \begin{pmatrix} \text{spot-next AUD Libor} \\ \text{1-week AUD Libor} \\ \text{2-week AUD Libor} \\ \text{1-month AUD Libor} \\ \text{2-month AUD Libor} \\ \vdots \\ \text{12-month AUD Libor} \end{pmatrix}. \qquad [0.3]$$

The value at time 3 of 1-month AUD Libor is $^{3}R_4$. The entire curve at time 1 is denoted $^{1}\boldsymbol{R}$. The univariate stochastic process representing 1-week AUD Libor over time is represented $\boldsymbol{R}_2$. The 15-dimensional stochastic process representing the entire curve over time is denoted $\boldsymbol{R}$. Time 0 is generally considered the current time. At time 0, current and past Libor curves are known. As non-random quantities, they are represented with lowercase letters: $\ldots, {}^{-3}\boldsymbol{r}, {}^{-2}\boldsymbol{r}, {}^{-1}\boldsymbol{r}, {}^{0}\boldsymbol{r}$. If time is measured in days, yesterday's value of 12-month AUD Libor is denoted $^{-1}r_{15}$.

The advantage of using preceding superscripts to denote time is clarity. By keeping time and component indices physically separate, my notation ensures one will never be confused for the other. Use of preceding superscripts is unconventional, but not without precedent. Actuarial notation makes extensive use of preceding superscripts.

Much other notation is standardized, as will become evident as the book unfolds. Frequently occurring notation is summarized in Exhibit 0.3. See Sections 1.8, 2.2, and 4.5 for more detailed explanations.

| | |
|---|---|
| *log* | natural logarithm |
| $n!$ | factorial of an integer $n$, which is given by the product: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$ |
| $U(a, b)$ | uniform distribution on the interval $(a, b)$ |
| $U_n(\Omega)$ | $n$-dimensional uniform distribution on the region $\Omega$ |
| $N(\mu, \sigma^2)$ | normal distribution with mean $\mu$ and variance $\sigma^2$ |
| $\Lambda(\mu, \sigma^2)$ | lognormal distribution with mean $\mu$ and variance $\sigma^2$ |
| $\chi^2(\nu, \delta^2)$ | chi-squared distribution with $\nu$ degrees of freedom and non-centrality parameter $\delta^2$ |
| $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ | joint-normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ |
| ${}^1P$ | random variable for a portfolio's value at time 1 |
| ${}^0p$ | portfolio value at time 0 |
| $({}^0p, {}^1P)$ | a portfolio |
| ${}^1L$ | random variable for portfolio loss: ${}^0p - {}^1P$ |
| ${}^1\boldsymbol{R}$ | random vector of key factors (key vector) |
| ${}^0\boldsymbol{r}$ | vector of key factor values at time 0 |
| ${}^1\boldsymbol{S}$ | random vector of asset values at time 1 (asset vector) |
| ${}^0\boldsymbol{s}$ | vector of asset values at time 0 |
| ${}^1\boldsymbol{Q}$, ${}^1\dot{\boldsymbol{Q}}$ or ${}^1\ddot{\boldsymbol{Q}}$ | frequently used to indicate risk vectors that are not key vectors |
| $E(\,)$ | unconditional expected value |
| ${}^tE(\,)$ | expected value conditional on information available at time $t$ |
| $std(\,)$ | unconditional standard deviation |
| ${}^tstd(\,)$ | standard deviation conditional on information available at time $t$ |
| $var(\,)$ | unconditional variance |
| ${}^tvar(\,)$ | variance conditional on information available at time $t$ |
| ${}^t\boldsymbol{\mu}$ | unconditional mean of the time $t$ term of a stochastic process |
| ${}^{t\|t-k}\boldsymbol{\mu}$ | mean of the time $t$ term of a stochastic process conditional on information available at time $t-k$ |
| ${}^t\boldsymbol{\Sigma}$ | unconditional covariance matrix of the time $t$ term of a stochastic process |

**Exhibit 0.3** Frequently used notation.

| | |
|---|---|
| ${}^{t\|t-k}\boldsymbol{\Sigma}$ | covariance matrix of the time $t$ term of a stochastic process conditional on information available at time $t-k$ |
| $\theta$ | frequently used to denote a portfolio mapping function |
| $\varphi$ | frequently used to denote a (non-portfolio) mapping function |
| $\boldsymbol{\omega}$ | portfolio holdings |
| ${}^{t}\phi(\ )$ | unconditional PDF of the time $t$ term of a stochastic process |
| ${}^{t\|t-k}\phi(\ )$ | PDF of the time $t$ term of a stochastic process conditional on information available at time $t-k$ |
| ${}^{t}\Phi(\ )$ | unconditional PDF of the time $t$ term of a stochastic process |
| ${}^{t\|t-k}\Phi(\ )$ | PDF of the time $t$ term of a stochastic process conditional on information available at time $t-k$ |
| $\sim$ | sometimes placed above notation to indicate a remapping; for example, ${}^{1}\widetilde{P} = \widetilde{\theta}({}^{1}\widetilde{\boldsymbol{R}})$ denotes a remapping of ${}^{1}P = \theta({}^{1}\boldsymbol{R})$. |
|  | indicates that analytic software more sophisticated than a spreadsheet may be useful in solving an exercise |

**Exhibit 0.3** *Continued.*

## 0.6. ACKNOWLEDGMENTS

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